ON THE INVARIANCE OF ORDER FOR FINITE-TYPE ENTIRE FUNCTIONS

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ABSTRACT. Let $f : \mathbb{C} \to \mathbb{C}$ be an entire function that has only finitely many critical and asymptotic values. Up to *topological equivalence*, the function f is determined by combinatorial information, more precisely by an infinite graph known as a *line complex*. In this note, we discuss the natural question whether the order of growth of an entire function is determined by this combinatorial information.

In particular, we provide various conditions that imply a positive answer to this question; including the *area property*, which is related to a conjecture of Eremenko and Lyubich. On the other hand, we discuss evidence that invariance of order and the area property fail in general.

1. Introduction

The *order* $\rho(f)$ of a meromorphic function $f : \mathbb{C} \to \hat{\mathbb{C}}$ is an important quantity in classical value distribution theory [18]. In the special case where $f : \mathbb{C} \to \mathbb{C}$ is an entire function, the order can be defined as

$$\rho(f) := \limsup_{z\to\infty} \frac{\log_+ \log_+ |f(z)|}{\log |z|} \in [0, \infty]$$

(where $\log_+ x = \max(0, \log x)$). Any polynomial or rational function has order 0, but there are also many transcendental entire and meromorphic functions with this property. On the other hand, the maximum modulus of an entire function can grow arbitrarily quickly; in particular, there are many functions of *infinite order*.

The set $S(f)$ of *singular values* of an entire function f is the smallest closed set $S \subset \mathbb{C}$ such that

$$f : f^{-1}(\mathbb{C} \setminus S) \to \mathbb{C} \setminus S$$

is a covering map. (Equivalently, $S(f)$ is the closure of the set of all critical and asymptotic values of f.) This set is of vital importance for both the function-theoretical and dynamical study of transcendental entire (and meromorphic) functions.

It is a guiding principle of both 1-dimensional holomorphic dynamics and 3-dimensional hyperbolic geometry that combinatorics determines geometry, under suitable finiteness assumptions. In this note, we consider a potential extension of this principle to value-distribution theory that was first proposed by the first author over fifteen years ago, but has not so far been discussed in print. The natural setting for our considerations is the *Speiser class*

$$\mathcal{S} := \{f : \mathbb{C} \to \mathbb{C} \text{ transcendental, entire} : S(f) \text{ is finite}\},$$

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which has been extensively studied both in dynamics and function theory. We shall also refer to such functions as *finite type maps*, in adopting terminology standard in holomorphic dynamics. We caution that the word *type* has an entirely different meaning in value-distribution theory. Recall that $\rho(f) \geq 1/2$ for every $f \in \mathcal{S}$, by the Ahlfors distortion theorem.

To a function $f \in \mathcal{S}$, one can associate an infinite planar graph, known as the *line complex*, which encodes the topological mapping behaviour of f. (We shall not use line complexes directly in this note, and hence refer the reader to [12] for further information.) From this combinatorial data, we can reconstruct the function f, up to pre- and post-composition by homeomorphisms. That is, the line complex determines f up to *topological equivalence* in the sense of Eremenko and Lyubich:

1.1. Definition (Topological equivalence)**.**
Two entire functions f and g are called *topologically equivalent* if there are homeomorphisms φ and ψ such that $\psi \circ f = g \circ \varphi$.

It is natural to ask which properties of f are combinatorially determined, and, in particular, whether this is the case for the order:

1.2. Question (Invariance of order)**.**
Let $f \in \mathcal{S}$ with $\rho(f) < \infty$, and let g be topologically equivalent to f. Is $\rho(f) = \rho(g)$?

For transcendental *meromorphic* functions, the order is given by

$$\rho(f) = \limsup_{r\to\infty} \frac{\log T(r,f)}{\log r}$$

where $T(r,f)$ is the *Nevanlinna characteristic* of f. (We emphasize that knowledge of Nevanlinna theory will not be required for the remainder of the paper.) Question 1.2 is partly motivated by the fact that the answer is positive in an important, classical case: that of meromorphic functions with rational Schwarzian derivative. As we briefly discuss in Section 2, in this case the order can be directly determined from the combinatorial information of the function (more precisely, $\rho(f) = \ell$, where ℓ is the number of *logarithmic ends*). On the other hand, the answer to the Question 1.2 for *meromorphic* finite-type functions is negative by classical work of Künzi [14].

In addition to the Speiser class, the larger *Eremenko-Lyubich class*

$$\mathcal{B} := \{f : \mathbb{C} \to \mathbb{C} \text{ transcendental, entire} : S(f) \text{ is bounded}\}$$

has also been studied extensively in complex dynamics. In this class, there do exist cases where the order is not invariant under topological equivalence.

1.3. Theorem (Counterexamples in class $\mathcal{B}$)**.** *There exist two finite-order functions $f_1, f_2 \in \mathcal{B}$ such that f_1 and f_2 are topologically equivalent, but $\rho(f_1) \neq \rho(f_2)$.*

These examples arise from complex dynamics. Indeed, suppose that p_1 and p_2 are complex polynomials of degree d with connected Julia sets, and that p_1 and p_2 are topologically conjugate. Then we shall see that the corresponding *Poincaré functions* (extended linearizing maps) at a repelling periodic cycle belong to the class $\mathcal{B}$, and are

topologically equivalent. On the other hand, their order is determined by the multiplier of the periodic cycle, which may change under topological conjugacy.

Note that this construction cannot be extended to yield counterexamples to invariance of order in the class $\mathcal{S}$. Indeed, a Poincaré function is in $\mathcal{S}$ if and only if the corresponding polynomial is *postcritically finite*, but postcritically finite maps are rigid by Thurston's theorem [5]. We are able to give a purely function-theoretic explanation of this phenomenon.

The area property. The following result, which is a consequence of the celebrated Teichmüller-Belinski theorem, will allow us to verify invariance of order for certain functions $f \in \mathcal{S}$.

1.4. Theorem (Invariance of order and the area property)**.** *Let $f \in \mathcal{S}$, and suppose that*

$$\int_{f^{-1}(K)\setminus\mathbb{D}} \frac{\mathrm{d}x\,\mathrm{d}y}{|z|^2} < \infty \tag{1.1}$$

for every compact set $K \subset \mathbb{C} \setminus S(f)$.

Then the order of f is invariant under topological equivalence. (Here $\mathbb{D} = \{|z| < 1\}$ denotes the unit disk.)

The condition (1.1) means that $f^{-1}(K) \setminus \mathbb{D}$ has finite cylindrical area; i.e. area with respect to the conformal metric $ds = |dz|/|z|$ on the punctured plane $\mathbb{C}^*$. Note that this condition makes perfect sense not just for a function $f \in \mathcal{S}$, but also for general entire functions f, and in particular for $f \in \mathcal{B}$:

1.5. Definition (The area property)**.**
We say that an entire function f satisfies the *area property* if

$$\int_{f^{-1}(K)\setminus\mathbb{D}} \frac{\mathrm{d}x\,\mathrm{d}y}{|z|^2} < \infty \tag{1.2}$$

for every compact set $K \subset \mathbb{C} \setminus S(f)$.

If this property holds for every compact subset of the unbounded connected component of $\mathbb{C} \setminus S(f)$, we say that f satisfies the area property *near infinity*.

We note that the area property is related to the following conjecture of Eremenko and Lyubich [10, p. 1009]:

1.6. Conjecture. *Suppose that $f \in \mathcal{S}$ is such that, for some $R > 0$,*

$$\liminf_{r\to\infty} \frac{1}{\ln r} \int_{\{z\in\mathbb{C}:1\le|z|\le r \text{ and } |f(z)|\le R\}} \frac{\mathrm{d}x\,\mathrm{d}y}{|z|^2} > 0.$$

Then f has a finite asymptotic value.

The area property is often easy to verify, allowing us to establish a positive answer to Question 1.2 in such cases. In particular, we can show that the linearizers of polynomials with connected Julia set typically satisfy the area property, even when invariance of order fails:

1.7. Theorem (The area property for linearizers)**.** *Let f be the Poincaré function associated to a repelling periodic point of a polynomial p with connected Julia set. Then $f \in \mathcal{B}$ and*

$$\rho(f) = \frac{\log \deg(p)}{\log |\lambda|} < \infty,$$

where λ is the multiplier of the repelling cycle.

Furthermore, f always satisfies the area property near infinity, and f has the area property if and only if p does not have any Siegel disks.

In particular, if p is postcritically finite, then $f \in \mathcal{S}$ and the order of f is invariant under topological equivalence.

The preceding theorem provides examples of functions in the class $\mathcal{B}$ where the area property fails. These examples rely in an essential way on the fact that the singular set of f (which includes the boundary of any Siegel disk of p) disconnects the plane. So it is natural to ask whether the area property holds for all finite-order functions $f \in \mathcal{S}$, or whether it holds *near infinity* for all finite-order functions $f \in \mathcal{B}$.

We provide some evidence that this is not the case:

1.8. Theorem (Counterexamples to the area property near infinity)**.** *There exist an unbounded simply-connected domain V and a holomorphic universal covering map*

$$f : V \to \mathbb{C} \setminus \overline{\mathbb{D}} =: W$$

such that

$$\int_{f^{-1}(U)} \frac{\mathrm{d}x\,\mathrm{d}y}{|z|^2} = \infty$$

for every nonempty open set $U \subset W$. The map f has finite order; i.e.

$$\limsup_{z\to\infty} \frac{\log_+ \log |f(z)|}{\log |z|} < \infty.$$

If $f \in \mathcal{B}$ has finite order and $S(f) \subset \mathbb{D}$, then $f^{-1}(W)$ (with W as above) consists of finitely many domains V_k, and $f : V_k \to W$ is a universal covering map for each k. Hence, if the area property near infinity fails for f, then one of these restrictions satisfies the conclusion of Theorem 1.8. Conversely, it is shown in [21] that it is possible to approximate many such covering maps by entire functions in the class $\mathcal{B}$, but at present our approximation results do not offer sufficient control to construct a function $f \in \mathcal{B}$ for which the area property fails near infinity. However, we know of no reason to expect that functions in class $\mathcal{B}$ and class $\mathcal{S}$ cannot exhibit the same type of behaviour as our example. Hence we consider Theorem 1.8 to provide strong evidence that invariance of order does not hold in general.

1.9. Conjecture. *There exist entire functions $f, g \in \mathcal{S}$ such that f and g are topologically equivalent, but $\rho(f) \neq \rho(g)$.*

Subsequent work. While this article was being prepared, a proof of Conjecture 1.9 was announced by Chris Bishop.

Basic notation. We shall assume that the reader is familiar with basic facts from complex geometry [11], hyperbolic geometry [2] and the theory of quasiconformal maps [1].

Throughout the paper, the complex plane, the punctured plane, the Riemann sphere and the unit disk are denoted $\mathbb{C}$, $\mathbb{C}^*$, $\hat{\mathbb{C}}$ and $\mathbb{D}$, respectively.

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2. Maps in the Speiser and Eremenko-Lyubich class

Singular values. Let $f : \mathbb{C} \to \mathbb{C}$ be a transcendental entire function. A point $z \in \mathbb{C}$ is called a *regular value* if there is an open $U \ni x$ such that f maps each component of $f^{-1}(U)$ homeomorphically onto U. Otherwise z is called a (finite) *singular value*, and the set of all such singular values is denoted by $S(f)$. Note that, since the set of regular values is open, this coincides with the definition of $S(f)$ given in the introduction.

Denote the sets of *critical* and *asymptotic* values of f by

$$C(f) = \{x \in \mathbb{C} : x = f(w) \text{ for some } w \in \mathbb{C} \text{ with } f'(w) = 0\} \qquad \text{and}$$
$$A(f) = \left\{ \begin{array}{c} x \in \mathbb{C} : x = \lim_{t\to 1} f(\gamma(t)) \text{ for some path} \\ \gamma : [0,1) \to \mathbb{C} \text{ with } \gamma(t) \to \infty \text{ as } t \to 1 \end{array} \right\},$$

respectively. Then it follows from covering theory that

$$S(f) = \overline{C(f) \cup A(f)}.$$

(Clearly $C(f) \cup A(f) \subset S(f)$, and if x has a neighborhood not intersecting $C(f) \cup A(f)$, then x is a regular value by the monodromy theorem.)

Topological and quasiconformal equivalence. Note that all three sets, $C(f)$, $A(f)$ and $S(f)$, are defined topologically, and hence are preserved by topological equivalence.

2.1. Observation. *Suppose that f and g are topologically equivalent, say $\psi \circ f = g \circ \varphi$. Then $A(g) = \psi(A(f))$, $C(g) = \psi(C(f))$ and $S(g) = \psi(S(f))$.*

Recall the definition of the *Speiser class* and the *Eremenko-Lyubich class* from the introduction:

$$\mathcal{S} := \{f : \mathbb{C} \to \mathbb{C} \text{ transcendental, entire} : S(f) \text{ is finite}\};$$
$$\mathcal{B} := \{f : \mathbb{C} \to \mathbb{C} \text{ transcendental, entire} : S(f) \text{ is bounded}\}.$$

One of the key properties of the class $\mathcal{S}$ with respect to topological equivalence is that the maps φ and ψ in Definition 1.1 can always be chosen to be quasiconformal (see Proposition 2.3 (d) below). For functions with infinitely many singular values, this need no longer be true, and it makes sense to introduce the following definition (see [20]):

2.2. Definition.
Two entire functions f and g are called *quasiconformally equivalent* if there are quasiconformal homeomorphisms $\varphi, \psi : \mathbb{C} \to \mathbb{C}$ such that $\psi \circ f = g \circ \varphi$.

We shall refer to φ and ψ from this definition or from Definition 1.1 as *witnessing homeomorphisms*.

The following facts regarding topological and quasiconformal equivalent are mostly folklore, and we shall provide the short proofs for completeness.

2.3. Proposition. (a) *Suppose that f and g are topologically equivalent, with witnessing homeomorphisms φ and ψ. If $\tilde{\psi}:\mathbb{C}\to\mathbb{C}$ is a homeomorphism that is isotopic to ψ relative $S(f)$, then there exists a homeomorphism $\tilde{\varphi}$, isotopic to φ relative $f^{-1}(S(f))$, such that $\tilde{\varphi}$ and $\tilde{\psi}$ are also witnessing homeomorphisms for f and g. If $\tilde{\psi}$ is quasiconformal, respectively conformal, then $\tilde{\varphi}$ is also.*

(b) *If f and g are quasiconformally equivalent and f has finite positive order, then g also has finite positive order. More precisely,*

$$0<\frac{1}{K}\leq\frac{\rho(g)}{\rho(f)}\leq K<\infty,$$

where K is the quasiconstant of φ.

(c) *Suppose that f and g are quasiconformally equivalent and that the witnessing homeomorphism φ is asymptotically conformal near infinity (that is, $\varphi(z)\sim a\cdot z$ for some $a\in\mathbb{C}$ as $z\to\infty$). Then $\rho(f)=\rho(g)$.*

(d) *If $f,g\in\mathcal{S}$ are topologically equivalent, then they are quasiconformally equivalent. If $\#S(f)=2$, then φ and ψ can be chosen to be affine (and $\rho(f)=\rho(g)$).*

Remark. It follows from the final statement in the theorem that the answer to Question 1.2 is always positive when $\#S(f)=2$.

Proof. The first statement follows by lifting the isotopy. More precisely, let ψ_t be an isotopy from ψ to $\tilde{\psi}$. Then, on every component of $U:=f^{-1}(\mathbb{C}\setminus S(f))$, there is a unique lift φ_t of this isotopy (since f is a covering map on each such component). So we have an isotopy $\varphi_t:U\to\varphi(U)$, and only need to show that the maps φ_t extend continuously to ∂U and agree with φ there. This is easily done using the continuity of f and ψ_t.

Indeed, we may assume without loss of generality that $\psi=\varphi=\text{id}$. Let $z_0\in\partial U$, and set $w_0:=f(z_0)$. Let U be a small neigborhood of z_0, chosen to ensure that $f:U\to V:=f(U)$ is a proper map with no critical points except possibly at z_0. We must show that $\varphi_t(z)\in U$ when z is sufficiently close to z_0. By continuity of f and the isotopy, if z is close enough to z_0, and $w:=f(z)$, then $\psi_t(w)\in V$ for all t. (Recall that $w_t:=\psi_t(w_0)=w_0$ for all t.) The point $z_t=\varphi_t(z)$ is obtained by analytic continuation of f^{-1} along the curve $\psi_t(w)$, which is entirely contained in V. Hence it follows that $z_t\in U$ for all t, as desired.

Away from the critical points of f, the homeomorphism φ can be written as a composition of ψ with an inverse branch of f^{-1}, hence if ψ is quasiconformal resp. conformal, then φ is also.

Claim (b) follows from the Hölder property of quasiconformal mappings: We have $|z|^{1/K}/C\leq|\varphi(z)|\leq C\cdot|z|^K$, and similarly for ψ. To use this in the formula for the order of g, let us write $z=\varphi(w)$. We have

$$\frac{\log_+\log_+|g(z)|}{\log|z|}=\frac{\log_+\log_+|\psi(f(w))|}{\log|\varphi(w)|}\leq\frac{\log_+\log_+C\cdot|f(w)|^K}{\log\frac{|w|^{1/K}}{C}}$$
$$=K\cdot\frac{O(1)+\log_+\log_+|f(w)|}{O(1)+\log|w|},$$

and hence $\rho(g) \leq K\rho(f)$. The opposite inequality follows analogously. Item (c) is immediate from the same computation.

For the final claim, we use the fact that, given witnessing homeomorphisms φ and ψ, we can find $\tilde{\psi}$ that is quasiconformal (resp. linear) and isotopic to ψ relative the finite set $S(f)$. Hence we can apply the first part of the Proposition and are done. ■

Maps with polynomial Schwarzian derivative: examples where the order is invariant. F. Nevanlinna's seminal work on the inverse problem of value-distribution theory involved a constructive study [17] of those meromorphic functions $f : \mathbb{C} \to \widehat{\mathbb{C}}$ whose *Schwarzian derivative* $\mathcal{S}_f = \left(\frac{f''}{f'}\right)' - \frac{1}{2}\left(\frac{f''}{f'}\right)^2$ is a polynomial. He characterized these maps as the meromorphic functions with finitely many "logarithmic ends", showing in particular that these are maps of finite type. For entire functions, the condition on $\mathcal{S}_f$ reduces to the requirement that f have polynomial *nonlinearity* $\mathcal{N}_f = \frac{f''}{f'}$.

Slightly more generally, G. Elfving [7] allowed finitely many critical points in addition to the finitely many logarithmic ends, to obtain the class of meromorphic functions $f : \hat{\mathbb{C}} \to \hat{\mathbb{C}}$ with rational Schwarzian derivative. (Compare also [8].) The finite poles of $\mathcal{S}_f$ are precisely the critical points of f, in fact

$$\mathcal{S}_f(\zeta) = \frac{m}{(z-\zeta)^2} + O\left(\frac{1}{z-\zeta}\right)$$

near a point ζ where $\deg_\zeta f = m$. The corresponding entire functions have rational nonlinearity, with

$$\mathcal{N}_f(\zeta) = \frac{m}{z-\zeta} + O(1)$$

near such a point ζ.

The number of logarithmic ends is a topological invariant, and hence does not change under topological equivalence. On the other hand, a calculation of asymptotics from the defining differential equations (see pp. 298-303 of [18], and pp. 391-393 of [13]) shows that this number coincides precisely with the order of the function f, as indicated in the introduction:

2.4. Proposition. *Let $f : \mathbb{C} \to \hat{\mathbb{C}}$ be a meromorphic function with rational Schwarzian derivative. Then $\rho(f) = \ell$ where $\ell = \deg_\infty \mathcal{S}_f$ is the number of logarithmic ends; in particular, if f is entire with rational nonlinearity then $\rho(f) = \deg \mathcal{N}_f$. Consequently, $\rho(f) = \rho(g)$ for any topologically equivalent map g.*

3. Poincaré functions: Counterexamples of bounded type

Let $h : \mathbb{C} \to \mathbb{C}$ be an entire function, and let $\zeta \in \mathbb{C}$ be a repelling fixed point of h. That is, $f(\zeta) = \zeta$ and $|\lambda| > 1$, where $\lambda = h'(\zeta)$ is the associated *multiplier*.

Then there exists a unique (up to restriction) conformal map f, defined near 0, such that $f(0) = \zeta$, $f'(0) = 1$ and

$$f(\lambda z) = h(f(z)). \tag{3.1}$$

(See e.g. [3, Theorem 6.3.2].) Using (3.1), we can extend f to an entire function $\mathbb{C} \to \mathbb{C}$ satisfying (3.1) for all $z \in \mathbb{C}$.

3.1. Definition (Poincaré function)**.**
A linearizing semiconjugacy $f : \mathbb{C} \to \mathbb{C}$ as above is called the *Poincaré function* of f at ζ.

If h is a polynomial of degree D, then it is easy to verify that f has finite order

$$\rho(f) = \frac{\log D}{\log |\lambda|} \tag{3.2}$$

(see [9]). Moreover, $S(f)$ coincides with the postcritical set $\mathcal{P}(p) = \overline{\bigcup_{n=1}^{\infty} p^n(C(p))}$ of p [16, Proposition 3.2]. In particular, f has finite type if and only if p is *postcritically finite*, and f belongs to the Eremenko-Lyubich class if and only if $\mathcal{P}(p)$ is bounded, which is equivalent to $J(p)$ being connected. Further function-theoretic properties of Poincaré functions have been investigated by Drasin and Okuyama [6].

To prove Theorem 1.3, we observe that a conjugacy between polynomials (and, in fact, entire functions) will result in the topological equivalence of their linearizers.

3.2. Proposition. *Suppose that f_1 and f_2 are non-constant, non-linear entire functions, and that f_1 and f_2 are topologically conjugate via a map $\psi : \mathbb{C} \to \mathbb{C}$; that is, $\psi \circ f_1 = f_2 \circ \psi$. Let x_1 be a repelling periodic point of f_1, set $x_2 := \psi(x_1)$, and let $L_1, L_2 : \mathbb{C} \to \mathbb{C}$ be the corresponding Poincaré functions of f_1 and f_2.*

Then there is a homeomorphism $\varphi : \mathbb{C} \to \mathbb{C}$ such that $\psi \circ L_1 = L_2 \circ \varphi$. If ψ is quasiconformal, then φ is also quasiconformal.

Proof. Let μ_2 be the branch of L_2 that takes x_2 to 0. We first define $\varphi(z)$, provided z is not a critical point of L_1. To do so, let α be a curve connecting 0 and z and not passing through any critical point of L_1. Then $\varphi(z)$ is obtained by analytic continuation of μ_2 along the curve $\psi \circ L_1 \circ \alpha$.

The fact that this analytic continuation is defined, and that it is independent of the curve α, can be seen as follows. Suppose that α_1 and α_2 are two different curves as above; then $\beta := \alpha_1 \cup \alpha_2$ is a closed curve beginning and ending at zero. Let us set $\gamma := \psi \circ L_1 \circ \beta$; this is a closed curve beginning and ending at x_2. We must show that there is a curve β_2, beginning and ending at zero, such that $L_2 \circ \beta_2 = \gamma$. To do so, let n be sufficiently large enough, and consider the curve

$$\gamma^n(t) := \psi(L_1(\beta(t)/\lambda_1^n)),$$

where λ_1 is the multiplier of f_1 at x_1. For sufficiently large n, the curve γ^n will be contained in a linearizing neighborhood of f_2 around x_2, so we can set $\beta_2^n := \mu_2 \circ \gamma^n$; this is a closed curve beginning and ending at zero. Set

$$\beta_2(t) := \lambda_2^n \cdot \beta_2^n(t),$$

where λ_2 is the multiplier of f_2 at x_2. Then $L_2 \circ \beta_2 = f_2^n \circ \gamma^n = \gamma$. Furthermore, since β does not contain any critical points of L_1, and ψ is a topological conjugacy (and hence sends critical points of f_1 to critical points of f_2), the curve β_2^n does not contain any critical points of f_2^n. Hence β_2 does not contain any critical points of L_2, as claimed.

This defines φ with the desired property on the complement of the set of critical points of L_1. It is easy to see (e.g. by applying the same construction, but reversing the roles of f_1 and f_2) that φ is a homeomorphism between the complement of the critical points

of L_1 and the complement of the critical points of L_2. Since both sets are discrete, it follows that φ extends to a homeomorphism $\varphi : \mathbb{C} \to \mathbb{C}$. (Alternatively, it is also easy to check directly that φ extends continuously to every critical point of L_1.)

If ψ is quasiconformal, then clearly φ is quasiconformal (as it is defined as a composition of locally quasiconformal maps). In this case (which is the one we are mainly interested in), there is an alternative and shorter proof of the proposition. Indeed, we can obtain the homeomorphism φ by solving a Beltrami equation for the pullback $L_1^*(\mu)$, where μ is the complex dilatation of ψ. Since μ is invariant under f_1, the pullback $L_1^*(\mu)$ is invariant under $z \mapsto \lambda_1 z$. It follows that φ conjugates $z \mapsto \lambda_1 z$ to a linear map, and hence that $g := \psi \circ L_1 \circ \varphi^{-1}$ semiconjugates f_2 to this linear map. Uniqueness of the Poincaré function implies $g = L_2$. ■

Proposition 3.2 implies Theorem 1.3, in the following stronger form:

3.3. Corollary. *There exist two functions $f_1, f_2 \in \mathcal{B}$ such that f_1 and f_2 are quasiconformally equivalent, but $\rho(f_1) \neq \rho(f_2)$.*

Proof. Consider the family of quadratic polynomials

$$p_\lambda : z \mapsto \lambda z + z^2,$$

with $0 < |2 - \lambda| < 1$. Then f_λ has a repelling fixed point of multiplier λ at 0, and an attracting fixed point of multiplier $2-\lambda$ at $1-\lambda$. It is well-known that any two elements of this family are quasiconformally conjugate.

Let L_λ be the Poincaré function for p_λ at zero, and set e.g. $f_1 := L_{3/2}$ and $f_2 := L_{4/3}$. Then f_1 and f_2 are quasiconformally equivalent by Proposition 3.2, while $\rho(f_1) \neq \rho(f_2)$ by (3.2). ■

4. The area property

The area property implies invariance of order. By the Teichmüller-Belinski theorem (see [15]), if $\varphi : \mathbb{C} \to \mathbb{C}$ is quasiconformal and

$$\lim_{R\to\infty} \int_{|z|>R} \left| \frac{\mu_\varphi(z)}{z^2} \right| \mathrm{d}x\, \mathrm{d}y = 0$$

then φ is asymptotically conformal at infinity. (Here μ_φ is the complex dilatation of φ.) This almost immediately leads to the proof of Theorem 1.4, which we state somewhat more generally:

4.1. Proposition (The area property and asymptotic conformality)**.** *Suppose that the entire functions f and g are quasiconformally equivalent, with witnessing homeomorphisms φ and ψ such that the dilatation of ψ is supported on a compact subset $K \subset \mathbb{C} \setminus S(f)$.*

If f satisfies the area property, then φ is asymptotically conformal near infinity, and hence $\rho(f) = \rho(g)$.

In particular, if f belongs to the class $\mathcal{S}$ and satisfies the area property, then $\rho(f) = \rho(g)$ for every function g that is topologically equivalent to f.

Proof. Since $g \circ \varphi = \psi \circ f$, and f and g are holomorphic, the dilatation of φ is supported on $f^{-1}(K)$. By the area property, this set has finite cylindrical area. Hence the Teichmüller-Belinski theorem implies that φ is indeed asymptotically conformal near infinity.

If furthermore $f \in \mathcal{S}$ and g is topologically equivalent to f, then, as discussed in Proposition 2.3 (d), the witnessing homeomorphisms φ and ψ can be chosen to be quasiconformal. Furthermore, let $\tilde{\psi}$ be a quasiconformal homeomorphism that is isotopic to ψ and whose dilatation is supported away from the singular values. Since $S(f)$ is finite, such a map clearly exists, and by Proposition 2.3 (a), there is $\tilde{\varphi}$ such that $\tilde{\psi}$ and $\tilde{\varphi}$ are witnessing homeomorphisms for the quasiconformal equivalence of f and g. So $\tilde{\varphi}$ is asymptotically conformal and $\rho(f) = \rho(g)$, as claimed. ∎

Some equivalent formulations of the area property. We shall now discuss some formulations of the area property that are easy to verify. We begin with an infinitesimal version:

4.2. Proposition (Infinitesimal area property)**.** *A function f has the area property if and only if*

$$\sum_{z \in f^{-1}(w) \setminus \mathbb{D}} \frac{1}{|z|^2 |f'(z)|^2} < \infty \tag{4.1}$$

for all $w \in \mathbb{C} \setminus S(f)$.

Furthermore, if (4.1) *holds for some $w_0 \in \mathbb{C} \setminus S(f)$, then it also holds for all w that belong to the same component of $\mathbb{C} \setminus S(f)$ as w_0.*

Proof. Let $w \in S(f)$, and let $D \subset \mathbb{C} \setminus S(f)$ be a closed topological disk whose interior contains w. Let $U \subset \mathbb{C} \setminus S(f)$ be a slightly larger simply-connected domain with $D \subset U$. By Koebe's distortion theorem [19, Theorem 1.3], every preimage component V of D is mapped to D by a conformal isomorphism with controlled distortion. It follows that the cylindrical area of V is proportional to $1/(|z|^2 |f'(z)|^2|)$, where z is the unique preimage of w in V.

If (4.1) holds and $K \subset S(f)$ is an arbitrary compact set, we can cover K by finitely many disks domains D_j as above, and the area property follows. Conversely, if (4.1) fails for some w, then we can take K to be a small closed disk around w.

The final claim follows by choosing D to contain both w_0 and w. ∎

There are various other ways to reformulate the area property. In particular, it is possible to reformulate (4.1) using hyperbolic geometry. Indeed, since f is a covering map on every component of $f^{-1}(\mathbb{C} \setminus S(f))$, the derivative $f'(z)$ can be expressed in terms of the hyperbolic metric of $f^{-1}(\mathbb{C} \setminus S(f))$. The hyperbolic metric of simply-connected domains is particularly easy to estimate in terms of the distance to the boundary, and hence we obtain the following.

4.3. Proposition (Distances and the area property)**.** *Let f be a transcendental entire function, and let $w \in \mathbb{C} \setminus S(f)$. Let $K \subset \mathbb{C} \setminus \{w\}$ be a closed connected set with $S(f) \subset K$ and $\#K > 1$. Then* (4.1) *holds if and only if*

$$\sum_{z \in f^{-1}(w) \setminus \mathbb{D}} \frac{\operatorname{dist}(z, f^{-1}(K))^2}{|z|^2} < \infty.$$

Proof. Let $z \in f^{-1}(w)$, let W be the component of $\mathbb{C} \setminus K$ containing w and let V be the component of $f^{-1}(W)$ containing z. Then $f : V \to W$ is a holomorphic covering map.

The domain W is either simply-connected or conformally equivalent to the punctured unit disk (if K is bounded and W is the unbounded connected component of $\mathbb{C} \setminus K$). The only covering spaces of the punctured disk are given by the universal covering (via the exponential map) and the punctured disk (via $z \mapsto z^d$, $d \geq 1$). The latter case cannot occur in our setting, since f is transcendental; so we see that V is simply-connected.

If ρ_V and ρ_W denote the densities of the hyperbolic metrics of V and W, we thus have $|f'(z)| = \rho_V(z)/\rho_W(w)$. Since

$$\frac{1}{2\,\mathrm{dist}(z, \partial V)} \leq \rho_V(z) \leq \frac{2}{\mathrm{dist}(z, \partial V)}$$

by the standard estimate on the hyperbolic metric in a simply-connected domain [2, Theorems 8.2 and 8.6], the claim follows. ■

A return to Poincaré functions. We now turn to studying the area property for Poincaré functions, proving Theorem 1.7.

4.4. Theorem (Area property for linearizers)**.** *Let p be a polynomial of degree ≥ 2 with a repelling fixed point at 0, and let $f : \mathbb{C} \to \mathbb{C}$ be the Poincaré function for this fixed point.*

Let $w \in \mathbb{C} \setminus \mathcal{P}(f) = \mathbb{C} \setminus S(f)$. Then (4.1) holds for w if and only if w does not belong to a Siegel disk of p.

Proof. It suffices to prove the theorem for $w \in F(f)$. Indeed, if $w \in J(f)$, then we can let w' be a point in the basin of infinity of p that belongs to the same component of $\mathbb{C} \setminus S(f)$ as w. By Proposition 4.2, property (4.1) holds for w' if and only if it holds for w.

If w does not belong to a Siegel disk, then it lies in the basin of infinity of p, an atttracting or parabolic basin, or a Fatou component that is not periodic. In each case, we can find a small disk D around w such that $p^{-n}(D) \cap D = \emptyset$ for all $n \geq 1$. Moreover, for any fixed δ, we may choose D sufficiently small to ensure that for every $n \geq 0$, every component of $p^{-n}(D)$ has diameter less than δ. (This follows from Koebe's theorem and the fact that all points of $p^{-n}(w)$ are close to the Julia set when n is sufficiently large. Recall that $S(f)$ agrees with the postcritical set of p.)

Let $\eta < 1$ be small enough so that f is injective on the disk of radius η around 0. We define

$$A := \{z \in \mathbb{C} : \frac{\eta}{2}|\lambda| < |z| < \eta/2\} \qquad \text{and} \qquad \tilde{A} := \{z \in \mathbb{C} : \frac{\eta}{4}|\lambda| < |z| < \eta\},$$

where $\lambda = p'(0)$. We assume that δ was chosen such that $\delta < \mathrm{dist}(f(A), \partial f(\tilde{A}))$; we shall estimate the cylindrical area of $f^{-1}(D)$.

Let $z \in f^{-1}(w) \setminus \mathbb{D}$ and let W_z be the component of $f^{-1}(D)$ containing z. Let $n = n_z$ be such that $\tilde{z} := z/\lambda^n \in \overline{A}$ and set $\tilde{w} := f(\tilde{z})$. Then $p^n(\tilde{w}) = w$ by the functional relation.

Let V_z be the component of $p^{-n}(D)$ containing $\tilde{w}$, and let $\tilde{W}_z$ be the component of $f^{-1}(V_z)$ containing $\tilde{z}$. By choice of δ and n, we have $V_z \subset f(\tilde{A})$ and hence $\tilde{W}_z \subset \tilde{A}$.

Clearly $\lambda^n \cdot \tilde{W}_z = W_z$; in particular, $\text{area}_{\mathbb{C}^*}(W_z) = \text{area}_{\mathbb{C}^*}(\tilde{W}_z)$ (Here $\text{area}_{\mathbb{C}^*}$ denotes cylindrical area.)

Let $z_1, z_2 \in f^{-1}(w) \setminus \mathbb{D}$ with $z_1 \neq z_2$. Then V_{z_i} is a component of $p^{-n_{z_i}}(D)$; hence $V_{z_1} \cap V_{z_2} = \emptyset$ by choice of D. So $\tilde{W}_{z_1} \cap \tilde{W}_{z_2} = \emptyset$, and hence (using Köbe's distortion theorem)

$$\begin{aligned}\text{const} \cdot \sum_{z \in f^{-1}(w)\setminus\mathbb{D}} \frac{1}{|z|^2|f'(z)|^2} &\leq \sum_{z \in f^{-1}(w)\setminus\mathbb{D}} \text{area}_{\mathbb{C}^*}(W_z) \\ &= \sum_{z \in f^{-1}(w)\setminus\mathbb{D}} \text{area}_{\mathbb{C}^*}(\tilde{W}_z) \leq \text{area}_{\mathbb{C}^*}(\tilde{A}) < \infty.\end{aligned}$$

On the other hand, suppose that w belongs to a Siegel disk U of p. Since $p|_U$ is conjugate to an irrational rotation, there is a sequence n_k such that $p_U^{n_k} \to \text{id}$.

Let $z \in f^{-1}(w)$ and set $z_k := \lambda^{n_k} \cdot z$. Then

$$z_k \cdot f'(z_k) = \frac{z_k}{\lambda^{n_k}} \cdot f'(z) \cdot (p^{n_k})'(w) \to z \cdot f'(z).$$

Thus

$$\sum_{z \in f^{-1}(w)\setminus\mathbb{D}} \frac{1}{|z|^2|f'(z)|^2} \geq \sum_{k=1}^{\infty} \frac{1}{|z_k|^2|f'(z_k)|^2} = \infty,$$

as required. ■

Remark. We did not use anywhere that p is a polynomial; hence the theorem holds also for a Poincaré function f of a transcendental entire function h, provided that the Fatou set of h is non-empty. It is plausible that the case where $J(h) = \mathbb{C}$ can be solved using the existence of *nice sets* D near any point of the Julia set that is not in the postsingular set [4]. Since these Poincaré functions will always have infinite order, we do not pursue the question further here.

5. Model counterexample to the area property near infinity

In this section, we prove Theorem 1.8. The simply-connected domain whose existence is asserted in the theorem is an instance of a general construction from [21]. Given a sequence

$$\Xi = (\varepsilon_k)_{k\in\mathbb{N}} \qquad \text{with } \varepsilon_k \in [0,1],$$

we define an associated simply-connected domain $V = V\big((\varepsilon_k)_{k\in\mathbb{N}}$. As shown in Figure 1, V consists of a central horizontal strip to which a sequence of chambers is attached, symmetrically with respect to the real axis. The k-th upper chamber is connected to the central strip by an opening of size proportional to ε_k, where $\varepsilon_k = 0$ means that the chamber is completely closed off, while $\varepsilon_k = 1$ means that the chamber is open.

More precisely:

5.1. Definition.
Set $w := 2\pi$ and $h := \pi/3$. For any sequence $\Xi = (\varepsilon_k)_{k\in\mathbb{N}}$ of numbers $\varepsilon_k \in [0,1]$, we

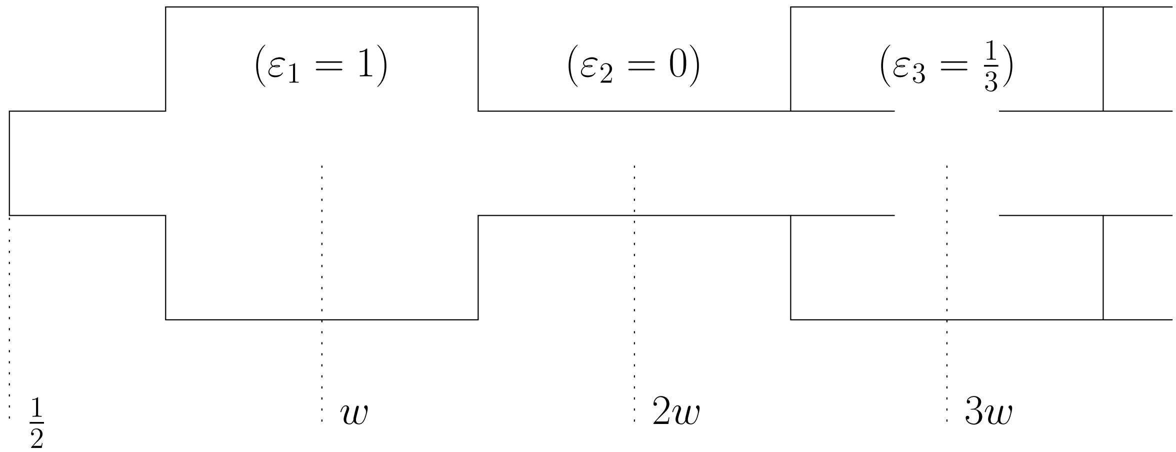

FIGURE 1. The domain $V = V\big((\varepsilon_k)_{k\in\mathbb{N}}\big)$

define

$$\begin{aligned} V := V\big(\Xi\big) := &\left\{a+ib : a > \frac{1}{2}, |b| < h\right\} \\ &\cup \bigcup_{k\in\mathbb{N}:\varepsilon_k\neq 0} \left\{a+ib : |a-kw| < \frac{w}{2}, h < |b| < \pi\right\} \\ &\cup \bigcup_{k\in\mathbb{N}} \left\{a+ib : |a-kw| < \frac{|\varepsilon_k|\cdot w}{2}, |b| = h\right\}. \end{aligned}$$

Furthermore, we define $F := F^{\Xi} : V \to \mathbb{H}$ be the unique conformal isomorphism with $F(1) = 1$ and $F'(1) > 0$, where $\mathbb{H} = \{x+iy : x > 0\}$ denotes the right half plane. Finally, we define

$$f := f^{\Xi} : \exp(V) \to W; \qquad f(e^z) := e^{F(z)},$$

where $W = \exp(\mathbb{H}) = \mathbb{C} \setminus \overline{\mathbb{D}}$.

5.2. Observation. *We have*

$$\int_{f^{-1}(U)} \frac{\mathrm{d}x\,\mathrm{d}y}{|z|^2} = \infty \tag{5.1}$$

for every nonempty open set $U \subset W$ if and only if

$$\sum_{k\in\mathbb{Z}} \mathrm{dist}(F^{-1}(w+2\pi ik), \partial V)^2 = \infty \tag{5.2}$$

for some $w \in \mathbb{H}$.

Proof. Define $\varphi := \exp\circ F$ and $\tilde{w} := \exp(w)$. By Koebe's theorem, the sum in (5.2) differs from

$$\sum_{k\in\mathbb{Z}} |(F^{-1})'(w+2\pi ik)|^2 = \frac{1}{|\tilde{w}|^2}\cdot \sum_{z\in\varphi^{-1}(\tilde{w})} \frac{1}{|\varphi'(z)|^2}$$

by at most a bounded factor; compare Proposition 4.3.

As in the proof of Proposition 4.2, Koebe's theorem implies that the convergence of the latter sum is independent of $\tilde{w}$, and is equivalent to the existence of an open set U containing $\tilde{W}$ such that $\varphi^{-1}(U)$ is finite. This area is precisely given by the integral in (5.1), and the proof is complete. ■

Thus, in order to prove Theorem 1.8, we should ensure that there is a suitable choice of Ξ such that the horocycle $\{z \in V : \operatorname{Re} F(z) = 1\}$ passes through infinitely many chambers in such a way as to stay away from the boundary.

Because the domain V is symmetric with respect to the real axis, we have $F([1,\infty)) = [1,\infty)$, and $F(\bar{z}) = \overline{F(z)}$. So it suffices to consider what happens for $k \geq 0$ in the sum (5.2).

5.3. Proposition. *If Ξ is suitably chosen, then there is a sequence $z_n \in V(\Xi)$ such that* $\operatorname{Re} z_n \to \infty$, $\operatorname{Re} F(z_n) = 1$ *and*

$$\limsup_{n\to\infty} \operatorname{dist}(z_n, \partial V) > c.$$

Proof. We begin with the sequence where $\varepsilon_k = 0$ for all k, and successively modify this sequence at values $k = k_n$. Let us set $\Phi^\Xi := (F^\Xi)^{-1}$.

For any sequence Ξ, and $k \in \mathbb{N}$, consider what happens when we change ε_k and keep all other values the same. When $\varepsilon_k = 0$, the horocycle $H = \Phi(\{\operatorname{Re} z = 1\})$ clearly does not enter the k-th chamber at all, while for $\varepsilon = 1$, it is easy to see that this curve stays close to the boundary of the chamber. Since the domain V depends continuously on Ξ (with respect to Carathéodory kernel convergence), there must be an intermediate value where the horocycle passes through the point $kw + 2ih$. That is, there is w_n with $\operatorname{Re} w_n = 1$ and $\Phi(w_n) = kw + 2ih$. Note that $\operatorname{dist}(\Phi(w_n), \partial V) = h$.

Now we construct Ξ inductively as follows. First choose some k_0 (e.g. $k_0 = 1$), and choose Ξ_0 with $\varepsilon_k = 0$ for $k \neq k_0$ as described above. Let w_0 be the corresponding point with $\operatorname{dist}(\Phi^{\Xi_0}(w_0), \partial V) = h$.

Now, if we choose $k = k_1$ sufficiently large, then—again by continuity in the Carathéodory topology—the value of $\Phi^\Xi(w_0)$ only moves by an arbitrarily small amount when we modify the k-th position in Ξ_0. So we can choose Ξ_1 and w_1 such that $\operatorname{dist}(\Phi^{\Xi_1}(w_1), \partial V) = h$ and

$$|\Phi^{\Xi_1}(w_0) - \Phi^{\Xi_0}(w_0)| < \frac{h}{4}.$$

In general, we choose Ξ_{n+1} such that

$$|\Phi^{\Xi_{n+1}}(w_j) - \Phi^{\Xi_n}(w_j)| < \frac{h}{2^{n+2}}$$

for $j \leq n$. For the limiting sequence Ξ, we then have

$$\operatorname{dist}(z_n, \partial V) \geq h/2,$$

where $z_n = \Phi^\Xi(w_n)$.

(This argument is carried out with more detail and some greater precision in [21, Theorem 7.4]. Note that, in that paper, the range of G is chosen to be a larger domain $H \supset \mathbb{H}$, but the proof remains the same.) ■

Proof of Theorem 1.8. Let Ξ, $V = V(\Xi)$ and $F = F^{\Xi}$ be as in the previous proposition, and let $z_n = F^{-1}(w_n)$ be the corresponding sequence. Let $m_n \in \mathbb{Z}$ be maximal such that $\operatorname{Im} w_n \geq 2\pi m_n$ and set $\tilde{w}_n := 1 + 2\pi i m_n$. Since the hyperbolic distance in $\mathbb{H}$ between w_n and $\tilde{w}_n$ is uniformly bounded, and since F is a conformal isomorphism, it follows that

$$\operatorname{dist}(F^{-1}(\tilde{w}_n, \partial V) \geq c',$$

for a suitable constant c'. In particular,

$$\sum_{k\in\mathbb{Z}} \operatorname{dist}(F^{-1}(2\pi i k), \partial V)^2 = \infty,$$

proving (5.1).

To see that f has finite order, first observe that the order of f is finite along the real axis. Indeed, the fact that the domain V contains a central strip of definite height implies (using the standard estimate on the hyperbolic metric) that

$$\log F(x) < \text{const} \cdot x$$

for all $x \geq 1$. To see that f has finite order overall, we observe that, for each k, the piece of V between $\frac{2k-1}{2}w$ and $\frac{2k+1}{2}w$ is a quadrilateral of uniformly bounded modulus. For details, we refer to [21, Lemma 7.2] ∎

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